0.1 Preface

This textbook is intended for high school students who are preparing for the Advanced Placement Exam in Calculus (AB or BC).

In this book, we assume you are familiar with high school algebra, analytic (Cartesian) geometry and graphing in the xy-plane, basic properties of trigonometric (trig) functions (in degrees), and basic properties of exponents and logarithms. We will define circular trig functions and radians in Section 2.7, define logarithms in Section 2.5, and define inverse trig functions in Section 2.9. However, in these sections, we will concentrate on the derivatives and graphs of the functions, not on algebraic properties.

We should remark on our approach to defining the exponential function in Section 2.4. There is always some difficulty in defining $\exp(x) = e^x$. If you already have integration, you can define the natural logarithm $\ln x$ via integration, and then define the exponential function as the inverse function of $\ln x$. This approach is elegant, but seems a bit "backwards" to most people, and would require developing integral Calculus before defining exp. This approach also makes it difficult to obtain bounds on the value of e. Then, there is the other calculator-based approach of "showing" that there is some number e, between 2 and 3, such that the limit, as h approaches zero, of $(e^h - 1)/h$ is equal to 1. The lack of rigor in this approach is worrisome and, once again, this approach makes it difficult to calculate bounds on the value of e.

We take a different approach from the two above. Our approach is via infinite series, a topic that is not covered until much later in this book. Consequently, we do not give a rigorous proof that our approach to defining e^x "works", until long after we have used e^x in many formulas and applications. We believe that there are several benefits to this series approach. First of all, we feel that students will have little trouble grasping that there is a sequence of polynomial functions such that the derivative of each element in the sequence is the previous element in the sequence (or zero), and that this sequence of functions can then be used to define a function which is its own derivative. Not only do we think that this approach poses no serious conceptual difficulty, but we hope that students will, in fact, find it "cool". Another advantage of using series to define $\exp(x)$ is that we can then show students how to calculate e, by hand, to any desired accuracy. A final advantage to our approach to exp is that we introduce students, briefly, to sequences, geometric series, and power series. We believe that this quick brush with sequences and series will make students more comfortable when they look at these concepts in detail later.

Our discussion of definite integrals, and their applications, is fairly traditional. However, our approach to infinite series is somewhat unusual. Our approach is motivated by two factors. First, we believe that the primary use that students will have for infinite series, outside of a Calculus class, is that many important functions have convergent power series representations, and these power series representations allow the student to mathematically manipulate and estimate the functions involved, in ways that would be difficult/impossible without power series. Second, statistical data that we collected over several years has made it clear that, in general, students do not grasp the basic idea that, when x is close to zero, smaller powers of x are more significant than larger powers of x in a power series or, even, in a polynomial function.

Consequently, we place emphasis on polynomial approximations and power series representations for functions, and, in a sense, view the classic convergence tests for sequences and series of constants as the "technical details" required to understand power series. We still include a chapter, Chapter 8, on sequences and series of constants, but that chapter comes **after** Chapter 7, which is on power series and approximating functions with polynomials. We firmly believe that this ordering of topics is better for the student and for applications, even though it may seem a bit awkward not to have the rigorous mathematical foundations of sequences and series come before their use in discussing power series.

Occasionally, when looking at approximations, we write an equals sign in quotes, as in "=". We use this to denote "equal as far as a calculator is concerned", i.e., equal to the precision of many/most/all calculators.

This book is organized as follows:

Each section is accompanied by a video link. Each video contains a classroom lecture of the essential contents of that section; if the student would prefer not to read the section, he or she can receive the same basic content from the video. The answers to most of the odd-numbered exercises are contained in Appendix D at the end of the book.

Important definitions are boxed in green, important theorems are boxed in blue. Remarks, especially warnings of common misconceptions or mistakes, are shaded in red. Important conventions, that will be used throughout the book, are boxed in black.

Occasionally, we refer to external sources for results beyond the scope of this textbook; our favorite external technical source is the excellent textbook by William F. Trench, *Introduction to Real Analysis*, [3], which is available as a free pdf.

Internal references through the text are hyperlinked; simply click on the boxed-in link to go to the appropriate place in the textbook. If you have activated the "forward" and "back" buttons in your pdf-viewer software, clicking on the "back" button will return you to where you started before you clicked on the hyperlink.

Some terms or names are annotated; these are clearly marked in the margins by little blue "balloons". Comments will pop up when you click on such annotated items.

We sincerely hope that you find using our modern, multimedia textbook to be as enjoyable as using a mathematics textbook can be.

David B. Massey
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Chapter 1

Rates of Change and the Derivative

In this chapter, we discuss what the rate of change of one quantity with respect to another means. We use the intuitive notion of an *average rate of change* to lead us to a definition of the *instantaneous rate of change*: the *derivative*. The transition from the average rate of change to the instantaneous rate of change requires us to develop the idea of the *limit* of a function.

We also show that our mathematical definition of the instantaneous rate of change has many, or all, of the properties that you intuitively expect.

1.1 Average Rates of Change

The world around us is in a continual state of change. Positions of people or objects change with respect to time; this rate of change is called *velocity*. Velocities change with time; this rate of change is called *acceleration*. The radius of a balloon increases with respect to the volume of air blown into the balloon. Like many rates of change, this latter one has no name, and so we simply have to use the entire phrase "the rate of change of the radius of the balloon, with respect to volume". The price of a lobster changes, with respect to the weight of the lobster (usually with jumps at certain weights). The area of a flat television screen, either in "full-screen" 4:3 format or in "wide-screen" 16:9 format, changes, with respect to the diagonal length. The y-coordinate of the graph of a function $y = f(x)$ changes, with respect to the x-coordinate.

In this section, we will begin our mathematical discussion of how you calculate *average rates of change* (AROC's) when one quantity, such as position, velocity, radius, price, area, or the y-coordinate, depends on (i.e., is a function of) another quantity, such as time, volume, weight, length, or the x-coordinate. Our goal in the next section will be to use the notion of an AROC, developed in this section, to arrive at a reasonable definition of an *instantaneous rate of change* (IROC). The study of instantaneous rates of change is what Differential Calculus is all about.

Let's look at a quick, but fundamental example of the type of question that we want to address.

Example 1.1.1.

Suppose that a car is traveling down a straight road. At exactly noon, the driver notices that she passes a mile marker, mile marker 37 (measured from some important point 37 miles back). At exactly 12:02 pm, the driver notices that she passes mile marker 38. What was the velocity of the car during the two minutes from noon until 12:02 pm?

You should be asking "What do you mean by 'the velocity of the car'? Do you mean what would someone inside the car have seen on the speedometer at each moment during the two minutes, or do you simply mean that the car went exactly one mile in 1/30th of an hour, so that its velocity was

$$\frac{1 \text{ mile}}{1/30 \text{ hour}} = 30 \text{ miles/hour ?"}$$

The above example is intended to illuminate the difference between the instantaneous rate of change, the IROC, and the average rate of change, the AROC.

The velocity that you read on the speedometer is the IROC of the position, with respect to time; we shall discuss this concept in detail in Section 1.2, Section 1.4, and throughout much of the remainder of this book.This velocity is itself a function of time; at each time between $t = 0$ and $t = 1/30$, you can read the instantaneous velocity on the car's speedometer.

Suppose that we let $p(t)$ be the position of the car, measured in miles, as determined by the mile markers, at time t hours past noon. The 30 miles/hour that we calculated above is the AROC of the position, with respect to time, between times $t = 0$ and $t = 1/30$ hours. This is the average velocity of the car between times $t = 0$ and $t = 1/30$ hours. In terms of the position function, $p(t)$, the average velocity of the car between times $t = 0$ and $t = 1/30$ hours is

$$\frac{\text{change in } p(t)}{\text{change in } t} = \frac{p(1/30) - p(0)}{1/30 - 0} = \frac{38 - 37 \text{ miles}}{1/30 \text{ hours}} = 30 \text{ miles per hour.}$$

Note that knowing this average velocity does not, in any way, tell us what the speedometer of the car was reading at any time.

Remark 1.1.2. The average velocity of the car between two given times is the AROC of the position with respect to time. It is **NOT** the average of the velocities at the two given times, that is, you do not add the velocities at the two different times and divide by 2. This fairly subtle difference in language leads to a huge difference in what you are calculating.

The phrase "the change in" that occurs when discussing various quantities in Calculus comes up so often that it is convenient to use one symbol to denote it. As is common, we shall use the Greek letter Δ for "the change in", so that the change in the position of the car in Example 1.1.1 would be denoted by Δp or $\Delta p(t)$. Of course, you cannot calculate Δp without being told the starting time and ending time, and without knowing the positions of the car at those times.

Using Example 1.1.1 as a guide, we would like to give the definition of the average rate of change for an "arbitrary" function. Of course, we want to use functions that you put real numbers into and from which you get real numbers back, that is, we want to use functions of the following type:

Definition 1.1.3. *A* **real function** f *is a function whose domain and codomain are subsets of the set* $\mathbb{R}$ *of real numbers.*

(The term *codomain* may be unfamiliar to you. There is little harm done if you replace every occurrence of the word "codomain" with "range", which should be a familiar term. The range is the set consisting precisely of those values which are attained by the function; the codomain is allowed to contain "extra", unattained values.)

All functions used throughout this book, for which the domain and codomain are not explicitly given, are assumed to be real functions.

In light of Example 1.1.1, we make the following definition for any (real) function $y = f(x)$. We use $y = f(x)$, since x, y, and f seem to be the favorite, generic variable and function names. You could just as easily use $z = p(t)$, and make the corresponding changes below.

Definition 1.1.4. *Suppose that* a *and* b *are in the domain of* f*, and* $a < b$*. Then, the* **average rate of change (the AROC) of** f**, with respect to** x**, between** $x = a$ **and** $x = b$**, or on the interval** $[a, b]$*, is*

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}.$$

Remark 1.1.5. It is important to note that the units of the average rate of change of f, with respect to x, are the units of f divided by the units of x.

Example 1.1.6.

Suppose that a car is moving (in a direction designated as positive) along a straight road, and that, at times $t = 0$, 1, and 5 hours (measured from some initial starting time), the car is moving at 30, 60, and 40 miles per hour, respectively. What are the average accelerations of the car on the intervals $[0, 1]$, $[0, 5]$, and $[1, 5]$?

Acceleration means the rate of change of the velocity, with respect to time. So, the average acceleration is the AROC of the velocity, with respect to time. If we let $v(t)$ denote the velocity of the car, in mph, at time t hours, then the average acceleration is $\Delta v/\Delta t$.

Hence, on the interval $[0, 1]$, the average acceleration is

$$\frac{v(1) - v(0)}{1 - 0} = \frac{60 - 30}{1 - 0} = 30 \text{ mph/hr (or mi/hr}^2\text{)}.$$

On the interval $[0, 5]$, the average acceleration is

$$\frac{v(5) - v(0)}{5 - 0} = \frac{40 - 30}{5 - 0} = 2 \text{ mph/hr},$$

and, on the interval $[1, 5]$, the average acceleration is

$$\frac{v(5) - v(1)}{5 - 1} = \frac{40 - 60}{5 - 1} = -5 \text{ mph/hr}.$$

This negative average acceleration is an indication that the car decelerated.

Example 1.1.7.

What is the average rate of change of the area A, in square inches, of a widescreen 16:9 television screen, with respect to the diagonal length d, between $d = 32$ inches and $d = 40$ inches? Between $d = 40$ inches and $d = 52$ inches?

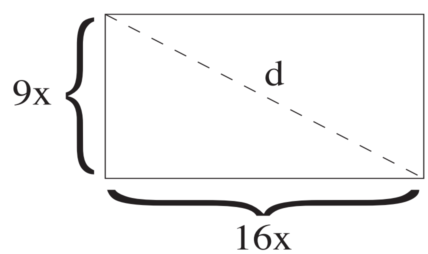

Figure 1.1: A widescreen 16:9 television.

The 16:9 ratio means that the width is 16/9 times the height. Suppose the height is $9x$. Then, the width is $16x$. The height, width, and diagonal measurements are related by the Pythagorean Theorem, so we have

$$(9x)^2 + (16x)^2 = d^2.$$

Also, we know that the area $A = (9x)(16x)$. It follows that $A = A(d) = 144d^2/337 \text{ in}^2$.

The AROC of the area, with respect to the diagonal length, on the interval $[32, 40]$, is

$$\frac{A(40) - A(32)}{40 - 32} = \frac{144}{337} \cdot \frac{40^2 - 32^2}{8} \approx 30.7656 \text{ in}^2/\text{in}.$$

The AROC of the area, with respect to the diagonal length, on the interval $[40, 52]$, is

$$\frac{A(52) - A(40)}{52 - 40} = \frac{144}{337} \cdot \frac{52^2 - 40^2}{12} \approx 39.3116 \text{ in}^2/\text{in}.$$

Example 1.1.8. Consider the function $y = f(x) = 4 - x^2$. What is the AROC of y, with respect to x, on the intervals $[1, 2]$ and $[1, 1.5]$?

We need to calculate $\Delta y/\Delta x$. On the interval $[1, 2]$, we find

$$\frac{\Delta y}{\Delta x} = \frac{f(2) - f(1)}{2 - 1} = \frac{0 - 3}{1} = -3.$$

On the interval $[1, 1.5]$, we find

$$\frac{\Delta y}{\Delta x} = \frac{f(1.5) - f(1)}{1.5 - 1} = \frac{1.75 - 3}{0.5} = -2.5.$$

The $\Delta y/\Delta x$ in Example 1.1.8 and in Definition 1.1.4 should remind you of the slope of a line, the *rise* over the *run*. Can we, in fact, picture the AROC of an arbitrary (real) function in terms of the slope of some line? Certainly. The AROC will be the slope of the line defined by:

Definition 1.1.9. *Given a function $y = f(x)$, and two x values a and b in the domain of f, with $a \neq b$, the* **secant line of f for $x = a$ and $x = b$** *is the line through the two points $(a, f(a))$ and $(b, f(b))$.*

Clearly, we have

Proposition 1.1.10. *Given a function $y = f(x)$, and $a < b$, where a and b are in the domain of f, the AROC of f on $[a, b]$ is equal to the slope of the secant line of f for $x = a$ and $x = b$.*

Example 1.1.11. Consider the function $y = f(x) = 4 - x^2$ from Example 1.1.8. The two AROC's on the intervals $[1, 2]$ and $[1, 1.5]$, calculated in Example 1.1.8, are the slopes of the two secant lines shown in Figure 1.2, on top of the graph of $y = 4 - x^2$.

The green line has slope equal to the AROC of f on the interval $[1, 2]$, which we already found to be -3. Using the point-slope form, an equation for this secant line is $y - 3 = -3(x - 1)$. The blue line has slope equal to the AROC of f on the interval $[1, 1.5]$, which we already found to be -2.5. An equation for this secant line is $y - 3 = -2.5(x - 1)$.

Example 1.1.12. Assume that we have an ideal balloon, which stays perfectly spherical as it inflates. What is the AROC of the radius of the balloon, with respect to the volume of air inside the balloon, as the volume changes from 20 in^3 to 30 in^3?

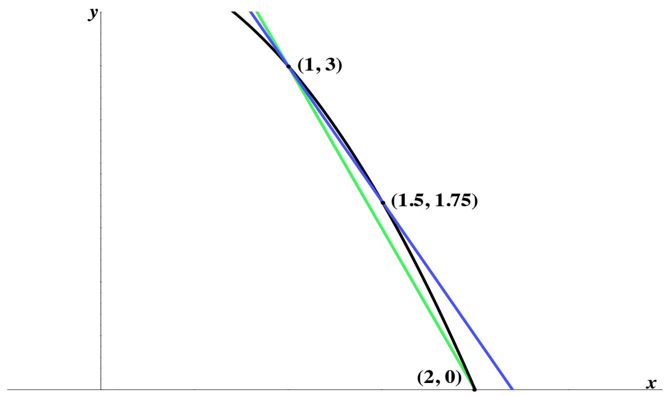

Figure 1.2: The graph of $y = 4 - x^2$ and two secant lines.

The volume of the balloon V, in in^3, is related to the radius R, measured in inches, by $V = (4/3)\pi R^3$. Thus, the radius of the balloon can be considered as a function of the volume of air in the balloon:

$$R = R(V) = \left(\frac{3V}{4\pi}\right)^{1/3} = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}.$$

The AROC of the radius with respect to volume on the interval $[20, 30]$ is

$$\frac{\Delta R}{\Delta V} = \frac{R(30) - R(20)}{30 - 20} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{(30)^{1/3} - (20)^{1/3}}{30 - 20} \approx 0.0243683 \text{ in/in}^3.$$

Example 1.1.13.

The price of lobsters per pound typically "jumps" at certain weights, to take into account the fact that a larger lobster has a smaller percentage of its weight contained in the shell. In addition, lobsters which weigh less than one pound are not sold.

Suppose that the price, in dollars per pound, for a lobster is given by $p = p(w)$, where w is the weight of the lobster in pounds, and $w \geq 1$. Let's assume that $p(w)$ is \$6/lb for $1 \leq w \leq 1.5$, \$7/lb for $1.5 < w \leq 2$, \$8/lb for $2 < w \leq 3$, and \$9/lb for $w > 3$. The total cost $C(w)$, in dollars, of a lobster is then equal to the number of pounds that the lobster weighs times the price per pound, i.e., $C(w) = w \cdot p(w)$. The graph of $C(w)$ versus w is given in Figure 1.3.

If we take the secant line between two points that lie on the same line segment in the graph, then the secant line will simply be the line containing the given line segment, and so the average rate of change of C, with respect to w, on the interval determined by the w values will simply be the slope of the corresponding line segment. For instance, if $2 < a < b \leq 3$, then the AROC of C, with respect to w, on the interval $[a, b]$ is

$$\frac{\Delta C}{\Delta w} = \frac{C(b) - C(a)}{b - a} = \frac{8b - 8a}{b - a} = \$8/\text{lb}.$$

Let's look at the AROC of C, with respect to w, between $w = 2$ and $w = 2.1$. We find

About the Author:

David B. Massey was born in Jacksonville, Florida in 1959. He attended Duke University as an undergraduate mathematics major from 1977 to 1981, graduating *summa cum laude*. He remained at Duke as a graduate student from 1981 to 1986. He received his Ph.D. in mathematics in 1986 for his results in the area of complex analytic singularities.

Professor Massey taught for two years at Duke as a graduate student, and then for two years, 1986-1988, as a Visiting Assistant Professor at the University of Notre Dame. In 1988, he was awarded a National Science Foundation Postdoctoral Research Fellowship, and went to conduct research on singularities at Northeastern University. In 1991, he assumed a regular faculty position in the Mathematics Department at Northeastern. He has remained at Northeastern University ever since, where he is now a Full Professor.

Professor Massey has won awards for his teaching, both as a graduate student and as a faculty member at Northeastern. He has published 34 research papers, and two research-level books. In addition, he was a chapter author of the national award-winning book on teaching: "Dear Jonas: What can I say?, Chalk Talk: E-advice from Jonas Chalk, Legendary College Teacher", edited by D. Qualters and M. Diamond, New Forums Press, (2004).

Professor Massey founded the Worldwide Center of Mathematics, LLC, in the fall of 2008, in order to give back to the mathematical community, by providing free or very low-cost materials and resources for students and researchers.